1 EXACT DIFFERENTIAL EQUATIONS

1.1. INTRODUCTION TO DIFFERENTIAL EQUATIONS

1.1.1. Differential Equation.

An equation involving independent variable, dependent variable and the derivatives of dependent variables with respect to independent variables is called a **differential equation.**

Mathematically, a differential equation can be expressed as

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots\ldots, \frac{d^ny}{dx^n}\right) = 0$$

Thus,
$$\frac{dy}{dx} = x \qquad \ldots(1)$$

$$\frac{dy}{dx} = \frac{1+y^2}{1+x^2} \qquad \ldots(2)$$

$$\frac{dy}{dx} = \tan x \qquad \ldots(3)$$

$$\left[1+\left(\frac{dy}{dx}\right)^3\right]^{4/3} = p\,.\,\frac{d^2y}{dx^2} \qquad \ldots(4)$$

$$\frac{d^3y}{dx^3} + 7\frac{d^2y}{dx^2} + 8\frac{dy}{dx} - 9y = \log x \qquad \ldots(5)$$

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0 \qquad \ldots(6)$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \qquad \ldots(7)$$

are all examples of differential equations.

1.1.2. Types of Differential Equations

Differential equations are of two types :

(*i*) *Ordinary differential equations*

(*ii*) *Partial differential equations*

(*i*) **Ordinary Differential Equations.** The differential equations which involve a single independent variable are called ordinary differential equations. In Art. 1.1.1., eqns. (1), (2), (3), (4), (5) are all ordinary differential equations.

(*ii*) **Partial Differential Equations.** The differential equations which involve partial differential co-efficients with respect to more than one independent variable are called partial differential equations. In Art. 1.1.1., equations (6) and (7) are partial differential equations.

In the present book, we shall be dealing only with ordinary differential equations.

1.1.3. Order and Degree of Differential Equations [M.D.U. 2013]

The order of a differential equation is the order of the highest differential co-efficient which occurs in it.

For example, equations (1), (2) and (3) of Art. 1.1.1. are of first order, equation (4) is of second order and equation (5) is of third order.

The ***degree*** *of a differential equation is the degree (or power) of the highest differential co-efficient which occurs in it after the differential equation has been cleared of radicals and fractions.*

For example, the equation $\left[1+\left(\frac{dy}{dx}\right)^3\right]^{4/3} = p\,\frac{d^2y}{dx^2}$, when freed from radicals and fractions takes the form

$$\left[1+\left(\frac{dy}{dx}\right)^3\right]^4 = p^3\left(\frac{d^2y}{dx^2}\right)^3.$$

Hence, the equation is of second order and third degree. In Art. 1.1.1., equations (1), (2), (3), (5), (6) and (7) all are of first degree.

1.1.4. Solution of a Differential Equation

A relation $y = f(x)$ between the dependent and independent variables, which when substituted in the equation, satisfies the equation is called a **solution** or **integral** of the given differential equation.

Generally, solution of differential equations are of three types :

(*a*) **General Solution.** The solution of differential equation in which the number of independent arbitrary constants (which cannot be reduced to a fewer number of equivalent constants) is the same as the order of the differential equation is called the general solution. It is also called *complete primitive or complete integral.*

(*b*) **Particular Solution.** The solution obtained by giving particular values to independent arbitrary constants in the general solution is called the *particular solution.*

(*c*) **Singular Solution.** A solution which does not contain any arbitrary constant and also, is not obtainable from the complete solution by giving particular values to the arbitrary constants is called a *singular solution.*

1.1.5. Formation of Differential Equations.

Formation of differential equations by elimination of arbitrary constants.

(*i*) Let the primitive be $f(x, y, a) = 0$ ...(1)

where '*a*' is an arbitrary constant.

Differentiating (1), we get $$F\left(x, y, \frac{dy}{dx}, a\right) = 0$$ ...(2)

Eliminating '*a*' between (1) and (2), we shall obtain a relation between x, y and $\frac{dy}{dx}$.

Let the relation obtained be $\psi\left(x, y, \frac{dy}{dx}\right) = 0$.

This is a differential equation of the first order. So we should expect one arbitrary constant in the solution of equations of first order.

(*ii*) Again consider the relation

$$f(x, y, a, b) = 0$$ ...(1)

By differentiating (1), suppose we get the relation

$$\phi_1\left(x, y, \frac{dy}{dx}, a, b\right) = 0$$ ...(2)

Differentiating (2) again, suppose we get

$$\phi_2\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, a, b\right) = 0$$ ...(3)

Eliminating '*a*', '*b*' between (1), (2) and (3), we shall get a relation of the form

$$\psi\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0,$$

which is a differential equation of second order. So we should expect two arbitrary constants in the solution of equations of second order.

1.5. SOLUTION OF AN EXACT DIFFERENTIAL EQUATION

[K.U. 2016, 14, 10; M.D.U. 2014, 13]

If the equation $M\,dx + N\,dy = 0$ is exact, then

$$M\,dx + N\,dy = d\left[u + \int f(y)\,dy\right]$$

But $$M\,dx + N\,dy = 0$$

$$\Rightarrow \qquad d\left[u + \int f(y)\,dy\right] = 0$$

Integrating both sides, we get

$$u + \int f(y)\,dy = c, \text{ where } c \text{ is an arbitrary constant.}$$

Putting $u = \int_{y\text{ constant}} M\,dx$, we get

$$\int_{y\text{ constant}} M\,dx + \int (\text{terms in N not containing } x)\,dy = c, \text{ which is the required solution.}$$

only value y

Working Rule :

(1) In an equation of the form $M\,dx + N\,dy = 0$, first check the condition of exactness

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

(2) Its solution is $\displaystyle\int_{y\text{ constant}} M\,dx + \int (\text{terms in N not containing } x)\,dy = c$

SOLVED EXAMPLES

Example 1.

Find the value of α, assuming that the differential equation

$$(1 + x^2y^3 + \alpha x^2y^2)\,dx + (2 + x^3y^2 + x^3y)\,dy = 0 \text{ is exact.}$$

Solution. The given differential equation is

$$(1 + x^2y^3 + \alpha x^2y^2)\,dx + (2 + x^3y^2 + x^3y)\,dy = 0 \qquad ...(1)$$

Comparing equation (1) with $M\,dx + N\,dy = 0$, we have

$$M = 1 + x^2y^3 + \alpha x^2y^2 \text{ and } N = 2 + x^3y^2 + x^3y$$

$$\therefore \quad \frac{\partial M}{\partial y} = 3x^2y^2 + 2ax^2y \quad \text{and} \quad \frac{\partial N}{\partial x} = 3x^2y^2 + 3x^2y$$

As the given differential equation is exact, so $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$\therefore \quad 3x^2y^2 + 2ax^2y = 3x^2y^2 + 3x^2y$$

$$\Rightarrow \quad 2a = 3 \quad \Rightarrow \quad a = \frac{3}{2}.$$

Example 2.

Solve the differential equation $(1 + e^{x/y})\, dx + e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0.$ [M.D.U. 2018]

Solution. The given differential equation is

$$(1 + e^{x/y})\, dx + e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0 \qquad \text{...(1)}$$

Comparing (1) with M dx + N dy = 0, we observe that

$$M = 1 + e^{x/y},$$

and
$$N = e^{x/y}\left(1 - \frac{x}{y}\right)$$

$$\therefore \quad \frac{\partial M}{\partial y} = e^{x/y}\left(-\frac{x}{y^2}\right) = -\frac{x}{y^2}\, e^{x/y}$$

and
$$\frac{\partial N}{\partial x} = e^{x/y} \cdot \frac{1}{y}\left(1 - \frac{x}{y}\right) + e^{x/y} \cdot \left(-\frac{1}{y}\right)$$

$$= e^{x/y}\left(\frac{1}{y} - \frac{x}{y^2} - \frac{1}{y}\right) = -\frac{x}{y^2}\, e^{x/y}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, therefore equation (1) is exact.

Hence the solution is $\displaystyle\int_{y \text{ constant}} M\, dx + \int (\text{terms in N not containing } x)\, dy = c$

i.e.,
$$\int_{y \text{ constant}} (1 + e^{x/y})\, dx = c$$

or
$$x + \frac{e^{x/y}}{1/y} = c \quad \Rightarrow \quad x + y\, e^{x/y} = c.$$

Example 3.

Solve the differential equation

$$(x^4 - 2xy^2 + y^4)\, dx - (2x^2y - 4xy^3 + \sin y)\, dy = 0. \quad \text{[M.D.U. 2013, 12, 2000]}$$

Solution. The given differential equation is

$$(x^4 - 2xy^2 + y^4)\, dx - (2x^2y - 4xy^3 + \sin y)\, dy = 0 \qquad ...(1)$$

Comparing equation (1) with $M\,dx + N\,dy = 0$, we observe that

$$M = x^4 - 2xy^2 + y^4$$

and $$N = (-2x^2y + 4xy^3 - \sin y)$$

$\therefore$ $$\frac{\partial M}{\partial y} = -4xy + 4y^3$$

and $$\frac{\partial N}{\partial x} = -4xy + 4y^3$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, therefore the given equation (1) is exact.

Hence the solution is

$$\int_{y\text{ constant}} M\,dx + \int (\text{terms in N not containing } x)\, dy = c$$

i.e., $$\int_{y\text{ constant}} (x^4 - 2xy^2 + y^4)\, dx + \int -\sin y\, dy = c$$

or $$\frac{x^5}{5} - 2y^2 \cdot \frac{x^2}{2} + y^4 \cdot x + \cos y = c$$

or $$\frac{x^5}{5} - x^2y^2 + xy^4 + \cos y = c.$$

Example 4.

Solve the differential equation

$$\frac{2x}{y^3}\, dx + \left(\frac{y^2 - 3x^2}{y^4}\right) dy = 0.$$

Solution. The given differential equation is

$$\frac{2x}{y^3}\, dx + \left(\frac{y^2 - 3x^2}{y^4}\right) dy = 0 \qquad ...(1)$$

Comparing equation (1) with M dx + N $dy = 0$, we observe that

$$M = \frac{2x}{y^3} \quad \text{and} \quad N = \frac{1}{y^2} - \frac{3x^2}{y^4}$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{-6x}{y^4} \quad \text{and} \quad \frac{\partial N}{\partial x} = -\frac{6x}{y^4}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, therefore the given equation (1) is exact.

Hence the solution is

$$\int_{y \text{ constant}} M\,dx + \int (\text{terms in N not containing } x)\,dy = c$$

i.e.,
$$\int_{y \text{ constant}} \frac{2x}{y^3}\,dx + \int \left(\frac{1}{y^2}\right) dy = c$$

or
$$\frac{2}{y^3} \int_{y \text{ constant}} x\,dx + \int \frac{1}{y^2}\,dy = c$$

or
$$\frac{x^2}{y^3} - \frac{1}{y} = c$$

or
$$x^2 - y^2 = cy^3.$$

Example 5.

Solve the differential equation $\left[\frac{y^2}{(y-x)^2}\,dx - \frac{x^2}{(x-y)^2}\,dy\right] = 0.$ [M.D.U. 2018]

Solution. The given differential equation is

$$\frac{y^2}{(y-x)^2}\,dx - \frac{x^2}{(x-y)^2}\,dy = 0 \qquad ...(1)$$

Comparing equation (1) with M dx + N $dy = 0$, we observe that

$$M = \frac{y^2}{(y-x)^2}$$

and
$$N = -\frac{x^2}{(x-y)^2}$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{(y-x)^2 \cdot 2y - y^2 \cdot 2(y-x)}{(y-x)^4}$$

$$= \frac{2y(y-x) - 2y^2}{(y-x)^3}$$

$$= \frac{-2xy}{(y-x)^3} = \frac{2xy}{(x-y)^3}$$

and
$$\frac{\partial N}{\partial x} = -\left[\frac{(x-y)^2 \cdot 2x - x^2 \cdot 2(x-y)}{(x-y)^4}\right]$$

$$= -\left[\frac{2x(x-y) - 2x^2}{(x-y)^3}\right] = \frac{2xy}{(x-y)^3}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, therefore the given equation (1) is exact.

Hence the solution is

$$\int_{y \text{ constant}} M\,dx + \int (\text{terms in N not containing } x)\,dy = c$$

$$\int_{y \text{ constant}} \frac{y^2}{(y-x)^2}\,dx - \int 0\,dy = c$$

$$y^2 \int_{y \text{ constant}} \frac{1}{(y-x)^2}\,dx = c$$

$$y^2 \int_{y \text{ constant}} (y-x)^{-2}\,dx = c$$

$$y^2 \cdot \frac{(y-x)^{-1}}{(-1)\times(-1)} = c$$

$$\frac{y^2}{y-x} = c \Rightarrow y^2 = c(y-x).$$

Example 6.

Verify that the differential equation $x\,dx + y\,dy = \frac{a^2(x\,dy - y\,dx)}{x^2+y^2}$ is exact and lve it.

Solution. The given differential equation is

$$x\,dx + y\,dy = \frac{a^2(x\,dy - y\,dx)}{x^2+y^2}$$

$$x\,dx + y\,dy = \frac{a^2 x}{x^2+y^2}\,dy - \frac{a^2 y}{x^2+y^2}\,dx$$

$$\left(x + \frac{a^2 y}{x^2+y^2}\right)dx + \left(y - \frac{a^2 x}{x^2+y^2}\right)dy = 0 \qquad ...(1)$$

Comparing equation (1) with M dx + N $dy = 0$, we observe that

$$M = x + \frac{a^2 y}{x^2 + y^2} \quad \text{and} \quad N = y - \frac{a^2 x}{x^2 + y^2}$$

$$\therefore \quad \frac{\partial M}{\partial y} = 0 + a^2 \left[\frac{(x^2 + y^2) - y\,(2y)}{(x^2 + y^2)^2} \right] = \frac{a^2(x^2 - y^2)}{(x^2 + y^2)^2}$$

and
$$\frac{\partial N}{\partial x} = 0 - a^2 \left[\frac{(x^2 + y^2) - x\,(2x)}{(x^2 + y^2)^2} \right] = \frac{a^2(x^2 - y^2)}{(x^2 + y^2)^2}$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, therefore equation (1) is exact.

Hence the solution is $\displaystyle\int_{y \text{ constant}} M\,dx + \int (\text{terms in N not containing } x)\, dy = c$

or
$$\int_{y \text{ constant}} \left(x + \frac{a^2 y}{x^2 + y^2} \right) dx + \int y\, dy = c$$

i.e.,
$$\frac{x^2}{2} + a^2 y \int_{y \text{ constant}} \frac{dx}{x^2 + y^2} + \frac{y^2}{2} = c$$

or
$$\frac{x^2}{2} + a^2 y \cdot \frac{1}{y} \tan^{-1} \frac{x}{y} + \frac{y^2}{2} = c \qquad \left[\because \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} \right.$$

or
$$x^2 + y^2 + 2a^2 \tan^{-1} \frac{x}{y} = 2c$$

or
$$x^2 + y^2 + 2a^2 \tan^{-1} \left(\frac{x}{y} \right) = K,$$

where $2c = K$ is any arbitrary constant.

EXERCISE 1.1

***Solve the following differential equations* [Q. 1 – 5] :**

1. (i) $(e^y + 1)\cos x\, dx + e^y \sin x\, dy = 0$

 (ii) $y \sin 2x\, dx - (y^2 + \cos^2 x)\, dy = 0$

 (iii) $[\cos x \tan y + \cos (x + y)]\, dx + [\sin x \sec^2 y + \cos (x + y)]\, dy = 0.$

2. (i) $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

 (ii) $\left[y\left(1 + \dfrac{1}{x}\right) + \cos y\right] dx + [x + \log x - x \sin y]\, dy = 0$

 (iii) $\cos x\, (\cos x - \sin a \sin y)\, dx + \cos y\, (\cos y - \sin a \sin x)\, dy = 0.$

3. (i) $x\, dx + y\, dy + \dfrac{x\, dy - y\, dx}{x^2 + y^2} = 0$ [M.D.U. 2012, 10

 (ii) $(x^2 - 2xy + 3y^2)\, dx + (4y^3 + 6xy - x^2)\, dy = 0$

 (iii) $(x^2 + y^2 + e^x)\, dx + 2xy\, dy = 0$

4. (i) $(2x^2y^3 + xy^2 + 3y)\,dx + (2x^3y^2 + x^2y + 3x)\,dy = 0$

(ii) $x(1+y^2)\,dx + y(1+x^2)\,dy = 0$.

(iii) $(a^2 - 2xy - y^2)\,dx - (x+y)^2\,dy = 0$ [M.D.U. 2001]

5. $(x^3 + 3xy^2)\,dx + (3x^2y + y^3)\,dy = 0$ [M.D.U. 2006]

ANSWERS

1. (i) $(e^y + 1)\sin x = c$ (ii) $\dfrac{y\cos 2x}{2} + \dfrac{y^3}{3} + c = 0$

(iii) $\sin x \tan y + \sin(x+y) = c$.

2. (i) $\tan y \tan x = c$ (ii) $xy + y\log x + x\cos y = c$

(iii) $2(x+y) + \sin 2x + \sin 2y - 4\sin a \sin y \sin x = K$.

3. (i) $x^2 + y^2 - 2\tan^{-1}\dfrac{x}{y} = K$ (ii) $\dfrac{1}{3}x^3 - x^2y + 3y^2x + y^4 = c$

(iii) $\dfrac{x^3}{3} + xy^2 + e^x = c$.

4. (i) $4x^3y^3 + 3x^2y^2 + 18xy = 6c$ (ii) $x^2 + (x^2+1)y^2 = 2c$

(iii) $a^2x - xy^2 - x^2y - \dfrac{y^3}{3} = c$ 5. $x^4 + 6x^2y^2 + y^4 = c$

1.6. INTEGRATING FACTOR [M.D.U. 2019]

Definition. *If a differential equation is not exact and it becomes exact after it has been multiplied by some suitable function of x and y, then such a function is called an* ***integrating factor***.

Integrating factor is denoted by **I.F.**

1.6.1. Theorem. Number of Integrating Factors.

To show that there is an infinite number of integrating factors for an equation $M\,dx + N\,dy = 0$.

Proof. Let μ be an integrating factor of $M\,dx + N\,dy = 0$ ...(1)

Then, for suitable function u of x and y, we have

$$\mu(M\,dx + N\,dy) = du \qquad ...(2)$$

$\therefore$ From (1) and (2), we have $du = 0$.

On integration, we have $u = c$, which is a solution of (2).

Multiplying (2) throughout by $f(u)$, a function of u, we get

$$\mu f(u)\,[M\,dx + N\,dy] = f(u)\,du$$

which is an exact differential equation.

$\therefore\ \mu f(u)$ is an I.F. of equation (2).

Since $f(u)$ is an arbitrary function of u, hence the number of integrating factors is infinite.

1.7. INTEGRATING FACTOR BY INSPECTION

Sometimes an integrating factor can be found by inspection. For this, the reader should study the following results :

	Group of Terms	*I.F.*	*Exact Differential*
1.	$x\,dy - y\,dx$	$\dfrac{1}{x^2}$	$\dfrac{x\,dy - y\,dx}{x^2} = d\left(\dfrac{y}{x}\right)$
2.	$y\,dx - x\,dy$	$\dfrac{1}{y^2}$	$\dfrac{y\,dx - x\,dy}{y^2} = d\left(\dfrac{x}{y}\right)$
3.	$x\,dy - y\,dx$	$\dfrac{1}{xy}$	$\dfrac{dy}{y} - \dfrac{dx}{x} = d\left(\log\dfrac{y}{x}\right)$
4.	$x\,dy - y\,dx$	$\dfrac{1}{x^2+y^2}$	$\dfrac{x\,dy - y\,dx}{x^2+y^2} = \dfrac{\dfrac{x\,dy - y\,dx}{x^2}}{1+\left(\dfrac{y}{x}\right)^2} = d\left(\tan^{-1}\dfrac{y}{x}\right)$
5.	$x\,dy + y\,dx$	$\dfrac{1}{(xy)^n}$	$\dfrac{x\,dy + y\,dx}{xy} = d\,[\log(xy)]$ for $n = 1$
6.	$x\,dx + y\,dy$	$\dfrac{1}{(x^2+y^2)^n}$	$\dfrac{x\,dx + y\,dy}{(x^2+y^2)^n} = d\left[\dfrac{-1}{2(n-1)(x^2+y^2)^{n-1}}\right]$ or $\dfrac{x\,dx + y\,dy}{x^2+y^2} = d\left[\dfrac{1}{2}\log(x^2+y^2)\right]$ if $n = 1$
7.	$ye^x\,dx - e^x\,dy$	$\dfrac{1}{y^2}$	$\dfrac{y\,e^x\,dx - e^x\,dy}{y^2} = d\left[\dfrac{e^x}{y}\right]$

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SOLVED EXAMPLES

Example 1.

Prove that $\frac{1}{x^2y^2}$ *is the integrating factor of the equation* [M.D.U. 2016]

$$(1 + xy)\, y\, dx + (1 - xy)\, x\, dy = 0$$

Solution. The given differential equation is

$$(1 + xy)\, y\, dx + (1 - xy)\, x\, dy = 0 \quad ...(1)$$

Comparing it with M dx + N dy = 0, we get

$$M = (1 + xy)\, y \quad \text{and} \quad N = (1 - xy)\, x$$

$$\therefore \quad \frac{\partial M}{\partial y} = 1 + 2xy \quad \text{and} \quad \frac{\partial N}{\partial x} = 1 - 2xy$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

∴ Equation (1) is not an exact equation.

Multiplying (1) by $\frac{1}{x^2y^2}$, we get

$$\frac{1 + xy}{x^2y}\, dx + \frac{1 - xy}{xy^2}\, dy = 0$$

or $$\left(\frac{1}{x^2y} + \frac{1}{x}\right) dx + \left(\frac{1}{xy^2} - \frac{1}{y}\right) dy = 0 \quad ...(2)$$

Comparing (2) with M dx + N dy = 0, we have

$$M = \frac{1}{x^2y} + \frac{1}{x} \quad \text{and} \quad N = \frac{1}{xy^2} - \frac{1}{y}$$

$$\therefore \quad \frac{\partial M}{\partial y} = -\frac{1}{x^2y^2} \quad \text{and} \quad \frac{\partial N}{\partial x} = -\frac{1}{x^2y^2}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

∴ Equation (2) is an exact equation.

Hence, $\frac{1}{x^2y^2}$ is an integrating factor of equation (1).

Example 2.

Find the value of α, if y^{α} is an integrating factor of the differential equation
$$2xy\,dx - (3x^2 - y^2)\,dy = 0.$$

Solution. The given differential equation is
$$2xy\,dx - (3x^2 - y^2)\,dy = 0 \quad ...(1)$$

It is given that y^{α} is an I.F. of (1), so multiplying (1) by y^{α}, we get
$$2xy\,y^{\alpha}\,dx - (3x^2 - y^2)\,y^{\alpha}\,dy = 0 \quad ...(2)$$

Comparing (2) with M dx + N dy = 0, we get
$$M = 2xy^{\alpha+1} \text{ and } N = -3x^2y^{\alpha} + y^{\alpha+2}$$
$$\therefore \quad \frac{\partial M}{\partial y} = (\alpha+1)\,2xy^{\alpha} \text{ and } \frac{\partial N}{\partial x} = -6xy^{\alpha}$$

As the given differential equation (2) is exact, so $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
$$\therefore \quad 2(\alpha+1)xy^{\alpha} = -6xy^{\alpha}$$
$$\Rightarrow \quad 2(\alpha+1) = -6 \Rightarrow \alpha+1 = -3 \Rightarrow \alpha = -4.$$

Example 3.

Show that $\frac{1}{x^2}$ is the integrating factor of the equation $(x^2+y^2)\,dx - 2xy\,dy = 0$. Also find its solution.

[K.U. 2019, 13]

Solution. The given differential equation is $(x^2+y^2)\,dx - 2xy\,dy = 0$ ...(1)

Comparing it with $Mdx + Ndy = 0$, we get
$$M = x^2 + y^2 \quad \text{and} \quad N = -2xy$$
$$\therefore \quad \frac{\partial M}{\partial y} = 2y \quad \text{and} \quad \frac{\partial N}{\partial x} = -2y$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, therefore equation (1) is not an exact equation.

Multiplying (1) by $\frac{1}{x^2}$, we have
$$\left(1 + \frac{y^2}{x^2}\right)dx - 2\frac{y}{x}\,dy = 0 \quad ...(2)$$

Comparing equation (2) with M dx + N dy = 0, we have
$$M = 1 + \frac{y^2}{x^2} \quad \text{and} \quad N = \frac{-2y}{x}$$
$$\therefore \quad \frac{\partial M}{\partial y} = \frac{2y}{x^2} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{2y}{x^2}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, therefore equation (2) is an exact equation.

Hence the solution is given by

$$\int_{y \text{ constant}} M\,dx + \int (\text{terms in N not containing } x)\,dy = c$$

i.e.,
$$\int_{y \text{ constant}} \left(1 + \frac{y^2}{x^2}\right) dx + 0 = c$$

i.e.,
$$x - \frac{y^2}{x} = c \quad \Rightarrow \quad x^2 - y^2 = cx.$$

Hence, I.F. $= \frac{1}{x^2}$ and the solution is $x^2 - y^2 = cx$.

Example 4.

Find the integrating factor and solve the differential equation

$$\boldsymbol{y\,(2xy + e^x)\,dx - e^x\,dy = 0.}$$

Solution. The given differential equation is $y\,(2xy + e^x)\,dx - e^x\,dy = 0$ ...(1)

Comparing it with Mdx + Ndy = 0, we have

$$M = 2xy^2 + ye^x \quad \text{and} \quad N = -e^x$$

$$\therefore \quad \frac{\partial M}{\partial y} = 4xy + e^x \quad \text{and} \quad \frac{\partial N}{\partial x} = -e^x$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, therefore equation (1) is not an exact equation.

Multiplying (1) by $\frac{1}{y^2}$, we get

$$\left(\frac{2xy + e^x}{y}\right) dx - \frac{e^x}{y^2}\,dy = 0$$

or
$$\left(2x + \frac{e^x}{y}\right) dx - \frac{e^x}{y^2}\,dy = 0 \quad \text{...(2)}$$

Comparing (2) with Mdx + Ndy = 0, we have

$$M = 2x + \frac{e^x}{y} \quad \text{and} \quad N = -\frac{e^x}{y^2}$$

$$\therefore \quad \frac{\partial M}{\partial y} = 0 - \frac{e^x}{y^2} \quad \text{and} \quad \frac{\partial N}{\partial x} = -\frac{e^x}{y^2}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, therefore equation (2) is an exact equation.

Hence the solution is $\int_{y \text{ constant}} M\,dx + \int (\text{terms in N not containing } x)\,dy = c$

i.e., $$\int \left(2x + \frac{e^x}{y}\right) dx = c$$

or $$x^2 + \frac{e^x}{y} = c$$

Hence I.F. $= \frac{1}{y^2}$ and the solution is $x^2 + \frac{e^x}{y} = c$.

Aliter : From equation (2) above, we have

$$2x\,dx + \frac{ye^x dx - e^x dy}{y^2} = 0$$

or $$2x\,dx + d\left(\frac{e^x}{y}\right) = 0$$

Integrating, we have $x^2 + \frac{e^x}{y} = c$, which is the required solution.

EXERCISE 1.2

Find the integrating factors and solve the following differential equations :

1. $y\,dx - x\,dy = 0$.
2. $y\,dx - x\,dy + \log x\,dx = 0$ [K.U. 2010]
3. $a\,(x\,dy + 2y\,dx) = xy\,dy$.
4. $(y^2e^x + 2xy)\,dx - x^2\,dy = 0$.

ANSWERS

1. $\frac{1}{xy}$ or $\frac{1}{y^2}$ or $\frac{1}{x^2}$; $y = cx$.
2. $\frac{1}{x^2}$; $cx + y + \log x + 1 = 0$.
3. $\frac{1}{xy}$; $2a \log x + a \log y - y = c$.
4. $\frac{1}{y^2}$; $e^x + \frac{x^2}{y} = c$.

1.8. RULES FOR FINDING THE INTEGRATING FACTORS

1.8.1. Rule 1. *If $M(x, y)$ and $N(x, y)$ are homogeneous functions in x, y and the equation $M\,dx + N\,dy = 0$ is not exact, then $\frac{1}{Mx + Ny}$ is an I.F., provided $Mx + Ny \neq 0$.*

Proof. The given equation is $Mdx + Ndy = 0$ ...(1)

As equation (1) is not exact, therefore we have to find the integrating factor.

SOLVED EXAMPLES

Example 1.

Find the integraing factor of the differential equation

$$(y^2 - 3xy)\,dx + (x^2 - xy)\,dy = 0.$$

Solution. The given differential equation is

$$(y^2 - 3xy)\,dx + (x^2 - xy)\,dy = 0 \quad ...(1)$$

Comparing equation (1) with M dx + N dy = 0, we have

$$M = y^2 - 3xy \text{ and } N = x^2 - xy$$

$$\therefore \quad \frac{\partial M}{\partial y} = 2y - 3x \text{ and } \frac{\partial N}{\partial x} = 2x - y$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, so the given equation (1) is not exact.

Now, $Mx + Ny = xy^2 - 3x^2 y + x^2y - xy^2 = -2x^2y \neq 0$

As equation (1) is homogeneous in x, y, therefore

$$\text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{-2x^2y}. \quad \text{[Using Rule I]}$$

Example 2.

Solve $x^2y\,dx - (x^3 + y^3)\,dy = 0.$

Solution. The given differential equation is $x^2y\,dx - (x^3 + y^3)\,dy = 0$ ...(1)

Comparing it with M dx + N dy = 0, we have

$$M = x^2y \quad \text{and} \quad N = -x^3 - y^3$$

$$\therefore \quad \frac{\partial M}{\partial y} = x^2 \quad \text{and} \quad \frac{\partial N}{\partial x} = -3x^2$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, therefore equation (1) is not exact.

As equation (1) is homogeneous in x and y, so

$$\text{I.F.} = \frac{1}{Mx + Ny}$$

Now, $$Mx + Ny = (x^2y)\,x + (-x^3 - y^3)\,y = -y^4 \neq 0$$

$$\therefore \quad \text{I.F.} = -\frac{1}{y^4}$$

Multiplying equation (1) by $-\frac{1}{y^4}$, we get

$$-\frac{x^2}{y^3}\,dx + \left(\frac{x^3}{y^4} + \frac{1}{y}\right)dy = 0 \quad \text{...(2)}$$

Now, equation (2) is an exact equation $\left[\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{3x^2}{y^4}\right]$

Hence the solution is $\int_{y\text{ constant}} M\,dx + \int (\text{terms in N not containing } x)\,dy = c,$

where c is any arbitrary constant.

i.e., $$\int_{y\text{ constant}} -\frac{x^2}{y^3}\,dx + \int \frac{1}{y}\,dy = c$$

or $$-\frac{1}{y^3}\int x^2\,dx + \int \frac{1}{y}\,dy = c$$

or $$-\frac{x^3}{3y^3} + \log y = c.$$

EXERCISE 1.3

Solve the following differential equations :

1. $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0.$ [K.U. 2014]
2. $(3xy^2 - y^3)\,dx - (2x^2y - xy^2)\,dy = 0.$ [K.U. 2015]
3. $(x^4 + y^4)\,dx - xy^3\,dy = 0.$
4. $y^2\,dx + (x^2 - xy - y^2)\,dy = 0.$

ANSWERS

1. $\frac{x}{y} - 2\log x + 3\log y = c.$
2. $3\log x + \frac{y}{x} - 2\log y = c.$
3. $y^4 = 4x^4 \log x + cx^4.$
4. $(x-y)\,y^2 = c\,(x+y).$

1.8.2. Rule 2. ***If the equation $M\,dx + N\,dy = 0$ is not exact and is of the form $f(xy)\,y\,dx + g(xy)\,x\,dy = 0$, then $\frac{1}{Mx - Ny}$ is an integrating factor provided $Mx - Ny \neq 0$.***

Proof. The given equation is $M\,dx + N\,dy = 0$ ...(1)

As equation (1) is not exact, therefore we have to find the I.F.

Suppose that $\frac{1}{Mx - Ny}$ is an I.F.

Multiplying eqn. (1) by I.F., we get

$$\frac{M}{Mx-Ny}\,dx+\frac{N}{Mx-Ny}\,dy=0$$

or
$$\frac{y\,f(xy)}{xy\,[f(xy)-g(xy)]}\,dx+\frac{x\,g(xy)}{xy\,[f(xy)-g(xy)]}\,dy=0$$

Now,
$$\frac{\partial}{\partial y}\left(\frac{y\,f(xy)}{xy\,[f(xy)-g(xy)]}\right)-\frac{\partial}{\partial x}\left(\frac{x\,g(xy)}{xy\,[f(xy)-g(xy)]}\right)$$

$$=\frac{\partial}{\partial y}\left(\frac{f(xy)}{x\,[f(xy)-g(xy)]}\right)-\frac{\partial}{\partial x}\left(\frac{g(xy)}{y\,[f(xy)-g(xy)]}\right)$$

$$=\left(\frac{[f(xy)-g(xy)]\,f'(xy)x-f(xy)\,[f'(xy)\,x-g'(xy)\,x]}{x\,[f(xy)-g(xy)]^2}\right)$$

$$-\left(\frac{[f(xy)-g(xy)]\,g'(xy)\,y-g(xy)\,[f'(xy)\,y-g'(xy)\,y]}{y\,[f(xy)-g(xy)]^2}\right)$$

$$=\frac{x\,[f(xy)\,g'(xy)-g(xy)\,f'(xy)]}{x\,[f(xy)-g(xy)]^2}-\frac{y\,[f(xy)\,g'(xy)-g(xy)\,f'(xy)]}{y\,[f(xy)-g(xy)]^2}$$

$$\therefore\quad \frac{\partial}{\partial y}\left(\frac{y\,f(xy)}{xy\,[f(xy)-g(xy)]}\right)-\frac{\partial}{\partial x}\left(\frac{x\,g(xy)}{xy\,[f(xy)-g(xy)]}\right)=0$$

Thus equation (2) is exact.

Hence to make equation (1) exact, $\dfrac{1}{Mx-Ny}$ is the I.F.

Example 1.

Solve $(xy^2+2x^2y^3)\,dx+(x^2y-x^3y^2)\,dy=0.$ [K.U. 2015, 12, 06; M.D.U. 2008, 07]

Solution. The given differential equation is

$$(xy^2+2x^2y^3)\,dx+(x^2y-x^3y^2)\,dy=0 \quad ...(1)$$

Comparing equation (1) with $M\,dx+N\,dy=0$, we observe that

$$M=xy^2+2x^2y^3 \quad\text{and}\quad N=x^2y-x^3y^2$$

$$\therefore\quad \frac{\partial M}{\partial y}=2xy+6x^2y^2 \quad\text{and}\quad \frac{\partial N}{\partial x}=2xy-3x^2y^2$$

Since $\dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}$, therefore the given equation (1) is not exact.

Also $\quad Mx - Ny = (xy^2 + 2x^2y^3)\,x - (x^2y - x^3y^2)\,y = 3x^3y^3 \neq 0.$

Equation (1) can be written as

$$(xy + 2x^2y^2)\,y\,dx + (xy - x^2y^2)\,x\,dy = 0 \qquad ...(2)$$

$\therefore$ Equation (2) is of the form

$$y\,f(xy)\,dx + x\,g(xy)\,dy = 0$$

$$\therefore \qquad \text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{3x^3y^3}$$

Multiplying equation (1) by $\dfrac{1}{3x^3y^3}$, we have

$$\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \left(\frac{1}{3xy^2} - \frac{1}{3y}\right)dy = 0,$$

which is an exact equation. $\left[\because \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} = -\dfrac{1}{3x^2y^2}\right]$

Hence the solution is $\displaystyle\int_{y\text{ constant}} M\,dx + \int (\text{terms in N not containing } x)\,dy = c_1,$

where c_1 is any arbitrary constant.

or $$\int_{y\text{ constant}} \left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \int -\frac{1}{3y}\,dy = c_1$$

or $$\frac{1}{3y}\int \frac{1}{x^2}\,dx + \frac{2}{3}\int \frac{1}{x}\,dx - \frac{1}{3}\int \frac{1}{y}\,dy = c_1$$

or $$-\frac{1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = c_1$$

or $$-\frac{1}{xy} + 2\log x - \log y = 3c_1$$

or $$-\frac{1}{xy} + 2\log x - \log y = c.$$

EXERCISE 1.4

Solve the following differential equations :

1. $(1 + xy)\,y\,dx + (1 - xy)\,x\,dy = 0.$
2. $(x^2y^2 + xy + 1)\,y\,dx + (x^2y^2 - xy + 1)\,x\,dy = 0.$ [*M.D.U. 2009, 2000; K.U. 2005*]
3. $(x^4y^4 + x^2y^2 + xy)\,y\,dx + (x^4y^4 - x^2y^2 + xy)\,x\,dy = 0.$ [*M.D.U. 2015*]
4. $(xy \sin xy + \cos xy)\,y\,dx + (xy \sin xy - \cos xy)\,x\,dy = 0.$ [*M.D.U. 2016*]

ANSWERS

1. $-\frac{1}{xy}+\log\frac{x}{y}=c$

2. $xy-\frac{1}{xy}+\log\frac{x}{y}=c$

3. $\frac{1}{2}x^2y^2-\frac{1}{xy}+\log\frac{x}{y}=c$

4. $x\sec(xy)=cy$ or $x=cy\cos(xy)$

1.8.3. Rule 3. *If the equation $M(x, y)dx+N(x, y)\,dy=0$ is not exact and $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}$ is a function of x only $=f(x)$ (say), then $e^{\int f(x)\,dx}$ is an integrating factor.*

Proof. The given equation is

$$M(x, y)\,dx + N(x, y)\,dy = 0 \quad \text{...(1)}$$

As equation (1) is not exact, therefore we have to find the integrating factor.

Suppose that $\mu(x)$ is an integrating factor, where $\mu(x)$ depends upon x only.

Multiplying equation (1) by $\mu(x)$, we get

$$\mu(x)\,M(x, y)\,dx + \mu(x)\,N(x, y)\,dy = 0$$

This equation is exact if and only if

$$\frac{\partial}{\partial y}[\mu(x)\,M(x, y)] = \frac{\partial}{\partial x}[\mu(x)N(x, y)]$$

or
$$\mu(x)\frac{\partial}{\partial y}[M(x, y)] = \mu(x)\frac{\partial}{\partial x}[N(x, y)] + N(x, y)\frac{\partial \mu(x)}{\partial x}$$

or
$$\frac{d}{dx}\mu(x) = \mu(x)\left[\frac{\dfrac{\partial}{\partial y}M(x, y)-\dfrac{\partial}{\partial x}N(x, y)}{N(x, y)}\right] \quad \text{...(2)}$$

$$\left[\frac{\partial \mu(x)}{\partial x}=\frac{d\mu(x)}{dx} \text{ as } \mu(x) \text{ is a function of } x \text{ only}\right]$$

Integrating (2), we get

$$\int\frac{d\mu(x)}{\mu(x)} = \int\frac{\dfrac{\partial}{\partial y}M-\dfrac{\partial}{\partial x}N}{N}\,dx$$

i.e.,
$$\log\mu(x) = \int\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}\,dx$$

$$\Rightarrow \qquad \mu(x) = e^{\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} dx}$$

$$\therefore \qquad \text{I.F.} = e^{\int \frac{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}{N} dx} = e^{\int f(x)\, dx}. \qquad \left[\because \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)\right]$$

SOLVED EXAMPLES

Example 1.

Find the integrating factor of the differential equation

$$\boldsymbol{2 \cos (y^2)\, dx - xy \sin (y^2)\, dy = 0.}$$

Solution. The given differential equation is

$$2 \cos (y^2)\, dx - xy \sin (y^2)\, dy = 0 \qquad \text{...(1)}$$

Comparing equation (1) with M dx + N dy = 0, we have

$$M = 2 \cos y^2 \quad \text{and} \quad N = -xy \sin y^2$$

$$\therefore \qquad \frac{\partial M}{\partial y} = -4y \sin y^2 \quad \text{and} \quad \frac{\partial N}{\partial x} = -y \sin y^2$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, therefore equation (1) is not exact.

Now, $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-4y \sin y^2 + y \sin y^2}{-xy \sin y^2} = \frac{3}{x} = f(x),$$

which is a function of x only.

$$\therefore \qquad \text{I.F.} = e^{\int f(x)\, dx} = e^{\int \frac{3}{x} dx} = e^{3 \log x} = x^3. \qquad [\textit{Using Rule } (3)]$$

Example 2.

Solve $\boldsymbol{(x^2 + y^2 + 2x)\, dx + 2y\, dy = 0.}$ [*K.U. 2013, 08; M.D.U. 2009, 07*]

Solution. The given differential equation is

$$(x^2 + y^2 + 2x)\, dx + 2y\, dy = 0. \qquad \text{...(1)}$$

Comparing (1) with M dx + N dy = 0, we have

$$M = x^2 + y^2 + 2x \quad \text{and} \quad N = 2y$$

$$\therefore \qquad \frac{\partial M}{\partial y} = 2y \qquad \text{and} \qquad \frac{\partial N}{\partial x} = 0$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, therefore equation (1) is not exact.

Now,
$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y - 0}{2y} = 1 = f(x)$$

$$\text{I.F.} = e^{\int f(x)\,dx} = e^{\int 1.dx} = e^x$$

$\therefore$ Multiplying equation (1) by e^x, we have

$$(x^2e^x + y^2e^x + 2xe^x)\,dx + 2ye^x\,dy = 0 \quad ...(2)$$

Now equation (2) is exact. $\left[\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2ye^x\right]$

Hence the solution is $\int_{y \text{ constant}} M\,dx + \int (\text{terms in N not containing } x)\,dy = c,$

where c is any arbitrary constant.

i.e.,
$$\int_{y \text{ constant}} (x^2e^x + y^2e^x + 2xe^x)\,dx = c$$

or
$$\int (x^2 + 2x)\,e^x\,dx + y^2 \int e^x\,dx = c$$

or
$$x^2e^x + y^2e^x = c. \quad \left[\because \int [f(x) + f'(x)]\,e^x\,dx = f(x)\,e^x\right]$$

Example 3.

Solve the differential equation

$$\boldsymbol{(2y \sin x + 3y^4 \sin x \cos x)\,dx - (4y^3 \cos^2 x + \cos x)\,dy = 0.}$$ [M.D.U. 201]

Solution. The given differential equation is

$$(2y \sin x + 3y^4 \sin x \cos x)\,dx - (4y^3 \cos^2 x + \cos x)\,dy = 0 \quad ...(1)$$

Comparing equation (1) with M dx + N dy = 0, we have

$$M = 2y \sin x + 3y^4 \sin x \cos x \quad \text{and} \quad N = -4y^3 \cos^2 x - \cos x$$

$$\therefore \quad \frac{\partial M}{\partial y} = 2 \sin x + 12y^3 \sin x \cos x \quad \text{and} \quad \frac{\partial N}{\partial x} = 8y^3 \cos x \sin x + \sin x$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, so equation (1) is not exact.

Now,
$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2 \sin x + 12y^3 \sin x \cos x - 8y^3 \cos x \sin x + \sin x}{-4y^3 \cos^2 x - \cos x}$$

$$= \frac{\sin x + 4y^3 \sin x \cos x}{-4y^3 \cos^2 x - \cos x}$$

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$$= \frac{\sin x\,(1+4y^3\cos x)}{-\cos x\,(1+4y^3\cos x)} = -\tan x = f(x)$$

which is a function of x only.

$$\therefore \quad \text{I.F.} = e^{\int f(x)\,dx}$$

$$= e^{-\int \tan x\,dx} = e^{\log \cos x} = \cos x$$

Multiplying equation (1) by $\cos x$, we get

$$(2y\sin x\cos x + 3y^4\sin x\cos^2 x)\,dx - (4y^3\cos^3 x + \cos^2 x)\,dy = 0 \quad ...(2)$$

Comparing (2) with $M\,dx + N\,dy = 0$, we have

$$M = 2y\sin x\cos x + 3y^4\sin x\cos^2 x \quad \text{and} \quad N = -4y^3\cos^3 x - \cos^2 x$$

$$\therefore \quad \frac{\partial M}{\partial y} = 2\sin x\cos x + 12y^3\sin x\cos^2 x \quad \text{and} \quad \frac{\partial N}{\partial x} = 12y^3\sin x\cos^2 x + 2\sin x\cos x$$

Here $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, therefore equation (2) is exact.

Hence the solution is

$$\int_{y\text{ constant}} M\,dx + \int (\text{terms in N not containing } x)\,dy = c$$

i.e.,
$$\int_{y\text{ constant}} (2y\sin x\cos x + 3y^4\sin x\cos^2 x)\,dx + \int 0\,dy = c$$

or
$$\frac{-2y\cos^2 x}{2} - \frac{3y^4\cos^3 x}{3} = c$$

or
$$y\cos^2 x + y^4\cos^3 x = c_1.$$

EXERCISE 1.5

Solve the following differential equations :

1. $(x^2+y^2+x)\,dx + xy\,dy = 0.$
2. $(xy^2 - e^{x^{1/3}})\,dx - x^2y\,dy = 0.$
3. $\left(y+\dfrac{y^3}{3}+\dfrac{x^2}{2}\right)dx + \dfrac{1}{4}(x+xy^2)\,dy = 0$ [K.U. 2017]
4. $(x^2+y^2+1)\,dx - 2xy\,dy = 0.$ [M.D.U. 2017, 14, 05]

ANSWERS

1. $3x^4 + 4x^3 + 6x^2y^2 = c$
2. $\dfrac{-y^2}{2x^2} + \dfrac{1}{3}e^{x^{1/3}} = c$
3. $3x^4y + x^4y^3 + x^6 = c$
4. $x^2 - y^2 = cx + 1.$

1.8.4. Rule 4. ***If the equation $M(x, y)\,dx + N(x, y)\,dy = 0$ is not exact and $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M}$ is a function of y only $= f(y)$ (say), then $e^{\int f(y)\,dy}$ is an integrating factor.***

Proof. The given equation is $M(x, y)\,dx + N(x, y)\,dy = 0$ ...(1)

As equation (1) is not exact, therefore we have to find the I.F.

Suppose that $\mu(y)$ is an integrating factor, where $\mu(y)$ depends upon y only.

Multiplying eqn. (1) by $\mu(y)$, we get $\mu(y) M\,dx + \mu(y) N\,dy = 0$ ...(2)

As eqn. (2) is exact, therefore

$$\frac{\partial}{\partial y}[\mu(y) M] = \frac{\partial}{\partial x}[\mu(y) N]$$

or $$M\frac{\partial}{\partial y}\mu(y) + \mu(y)\frac{\partial M}{\partial y} = \mu(y)\frac{\partial N}{\partial x}$$

or $$M\frac{d}{dy}\mu(y) + \mu(y)\frac{\partial M}{\partial y} = \mu(y)\frac{\partial N}{\partial x}$$

$\left[\textit{Here } \dfrac{\partial}{\partial y}\mu(y) = \dfrac{d}{dy}\mu(y) \textit{ as } \mu(y) \textit{ is a function of } y \textit{ only}\right.$

$$\Rightarrow \quad M\frac{d}{dy}\mu(y) = \mu(y)\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$$\Rightarrow \quad \frac{\dfrac{d}{dy}\mu(y)}{\mu(y)} = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \quad ...(3)$$

Integrating (3), we get $\displaystyle\int \frac{d\mu(y)}{\mu(y)} = \int \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy$

$$\Rightarrow \quad \log \mu(y) = \int \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy$$

$$\therefore \quad \mu(y) = e^{\int \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy} = e^{\int f(y)\,dy}$$, which is the required I.F.

SOLVED EXAMPLES

Example 1.

Solve $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0.$ *[M.D.U. 2013, 03; K.U. 2011, 0*

Solution. The given differential equation is

$$(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0 \quad ...(1$$

Comparing equation (1) with M dx + N dy = 0, we observe that

$$M = 3x^2y^4 + 2xy \quad \text{and} \quad N = 2x^3y^3 - x^2$$

$$\therefore \quad \frac{\partial M}{\partial y} = 12x^2y^3 + 2x \quad \text{and} \quad \frac{\partial N}{\partial x} = 6x^2y^3 - 2x$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, therefore equation (1) is not exact.

Now,
$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{6x^2y^3 - 2x - 12x^2y^3 - 2x}{xy(3xy^3 + 2)}$$

$$= -\frac{2x(3xy^3 + 2)}{xy(3xy^3 + 2)} = \frac{-2}{y} = f(y)$$

which is a function of y only.

$$\therefore \quad \text{I.F.} = e^{\int f(y)\,dy} = e^{\int -\frac{2}{y}dy} = e^{-2\log y} = e^{\log y^{-2}}$$

$$= e^{\log \frac{1}{y^2}} = \frac{1}{y^2}$$

Multiplying both sides of equation (1) by $\frac{1}{y^2}$, we get

$$\left(3x^2y^2 + \frac{2x}{y}\right)dx + \left(2x^3y - \frac{x^2}{y^2}\right)dy = 0 \qquad ...(2)$$

Now equation (2) is exact. $\left[\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 6x^2y - \frac{2x}{y^2}\right]$

Hence the solution is $\int_{y\text{ constant}} \left(3x^2y^2 + \frac{2x}{y}\right)dx + 0 = c$, where c is any arbitrary constant.

or
$$x^3y^2 + \frac{x^2}{y} = c$$

or
$$x^3y^3 + x^2 = cy.$$

EXERCISE 1.6

Solve the following differential equations :

1. $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0$
2. $(2xy^4e^y + 2xy^3 + y)\,dx + (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0$ [M.D.U. 2008]
3. $(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0$ [M.D.U. 2012]
4. $(xy^2 - x^2)\,dx + (3x^2y^2 + x^2y - 2x^3 + y^2)\,dy = 0.$ [M.D.U. 2013, 07; K.U. 2011]

ANSWERS

1. $xy + y^2 + \dfrac{2x}{y^2} = c$

2. $x^2e^y + \dfrac{x^2}{y} + \dfrac{x}{y^3} = c$

3. $3x^2y^4 + 6xy^2 + 2y^6 = c$

4. $e^{6y}\left(\dfrac{x^2y^2}{2} - \dfrac{x^3}{3} + \dfrac{y^2}{6} - \dfrac{y}{18} + \dfrac{1}{108}\right) = c.$

1.8.5. Rule 5. ***If the equation $Mdx + Ndy = 0$ can be expressed as $x^ay^b\,(my\,dx + nx\,dy) + x^cy^d\,(py\,dx + qx\,dy) = 0$, where a, b, c, d, m, n, p, q are constants and $\dfrac{m}{n} \neq \dfrac{p}{q}$, then $x^\alpha y^\beta$ is an integrating factor, when α and β are so chosen that $\dfrac{a+\alpha+1}{m} = \dfrac{b+\beta+1}{n}$ and $\dfrac{c+\alpha+1}{p} = \dfrac{d+\beta+1}{q}$***

Proof. The given differential equation is

$$x^ay^b\,(my\,dx + nx\,dy) + x^cy^d\,(py\,dx + qx\,dy) = 0 \quad \ldots(1)$$

Suppose $x^\alpha y^\beta$ is an integrating factor of (1).

Multiplying eqn. (1) by I.F., we get

$$(mx^{a+\alpha}\,y^{b+\beta+1}\,dx + nx^{a+\alpha+1}\,y^{b+\beta}\,dy) + (px^{c+\alpha}\,y^{d+\beta+1}\,dx + qx^{c+\alpha+1}\,y^{d+\beta}\,dy) = 0$$

or $$(mx^{a+\alpha}\,y^{b+\beta+1} + px^{c+\alpha}\,y^{d+\beta+1})\,dx + (nx^{a+\alpha+1}\,y^{b+\beta} + qx^{c+\alpha+1}\,y^{d+\beta})\,dy = 0$$

which is of the form $M\,dx + N\,dy = 0$.

Now,
$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{\partial}{\partial y}(mx^{a+\alpha}\,y^{b+\beta+1} + px^{c+\alpha}\,y^{d+\beta+1}) - \frac{\partial}{\partial x}(nx^{a+\alpha+1}\,y^{b+\beta} + qx^{c+\alpha+1}\,y^{d+\beta})$$

$$= [(b+\beta+1)\,mx^{a+\alpha}\,y^{b+\beta} + (d+\beta+1)\,px^{c+\alpha}y^{d+\beta}] - [(a+\alpha+1)\,ny^{b+\beta}\,x^{a+\alpha} + (c+\alpha+1)qy^{d+\beta}\,x^{c+\alpha}]$$

$$= [(b+\beta+1)\,m - (a+\alpha+1)n]\,y^{b+\beta}\,x^{a+\alpha} + [(d+\beta+1)\,p - (c+\alpha+1)q]\,y^{d+\beta}\,x^{c+\alpha}$$

$$= mn\left[\frac{(b+\beta+1)}{n} - \frac{(a+\alpha+1)}{m}\right]x^{a+\alpha}\,y^{b+\beta} + pq\left[\frac{(d+\beta+1)}{q} - \frac{(c+\alpha+1)}{p}\right]x^{c+\alpha}\,y^{d+\beta}$$

$$= 0 + 0 = 0 \qquad \left[\because \frac{a+\alpha+1}{m} = \frac{b+\beta+1}{n} \text{ and } \frac{c+\alpha+1}{p} = \frac{d+\beta+1}{q}\right.$$

$$\therefore \quad \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0$$

Thus, equation (2) is an exact differential equation.

Hence $x^\alpha y^\beta$ is an I.F., provided $\dfrac{a+\alpha+1}{m} = \dfrac{b+\beta+1}{n}$ and $\dfrac{c+\alpha+1}{p} = \dfrac{d+\beta+1}{q}$.

SOLVED EXAMPLES

Example 1.

Solve : $(2x^2y - 3y^4)\,dx + (3x^3 + 2xy^3)\,dy = 0.$ [K.U. 2018, 16; M.D.U. 2014, 11]

Solution. The given equation is

$$(2x^2y - 3y^4)\,dx + (3x^3 + 2xy^3)\,dy = 0. \quad ...(1)$$

Comparing equation (1) with $M\,dx + N\,dy = 0$, we observe that

$$M = 2x^2y - 3y^4 \quad \text{and} \quad N = 3x^3 + 2xy^3$$

$$\therefore \quad \frac{\partial M}{\partial y} = 2x^2 - 12y^3 \quad \text{and} \quad \frac{\partial N}{\partial x} = 9x^2 + 2y^3$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, therefore equation (1) is not exact.

Now equation (1) can be written as

$$x^2(2y\,dx + 3x\,dy) + y^3(-3y\,dx + 2x\,dy) = 0$$

which is of the form

$$x^a y^b (my\,dx + nx\,dy) + x^c y^d (py\,dx + qx\,dy) = 0,$$

where $a = 2,\ b = 0,\ m = 2,\ n = 3,\ c = 0,\ d = 3,\ p = -3$ and $q = 2.$

Let the integrating factor be $x^\alpha y^\beta$.

Multiplying equation (1) on both sides by the I.F. *i.e.*, $x^\alpha y^\beta$, we have

$$(2x^{\alpha+2}y^{\beta+1} - 3x^\alpha y^{\beta+4})\,dx + (3x^{\alpha+3}y^\beta + 2x^{\alpha+1}y^{\beta+3})\,dy = 0 \quad ...(2)$$

Now comparing equation (2) with $M\,dx + N\,dy = 0$, we have

$$M = 2x^{\alpha+2}y^{\beta+1} - 3x^\alpha y^{\beta+4}$$

and $$N = 3x^{\alpha+3}y^\beta + 2x^{\alpha+1}y^{\beta+3}$$

Then $$\frac{\partial M}{\partial y} = 2(\beta+1)x^{\alpha+2}y^\beta - 3(\beta+4)x^\alpha y^{\beta+3}$$

and $$\frac{\partial N}{\partial x} = 3(\alpha+3)x^{\alpha+2}y^\beta + 2(\alpha+1)x^\alpha y^{\beta+3}$$

Now equation (2) will be exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

i.e., if $$2(\beta+1)x^{\alpha+2}y^\beta - 3(\beta+4)x^\alpha y^{\beta+3} = 3(\alpha+3)x^{\alpha+2}y^\beta + 2(\alpha+1)x^\alpha y^{\beta+3}$$

or if $\quad 2(\beta+1) = 3(\alpha+3)$ [Comparing co-effs. of $x^{\alpha+2}y$

and $\quad -3(\beta+4) = 2(\alpha+1)$ [Comparing co-effs. of $x^{\alpha}y^{\beta+}$

or if $\quad \alpha = -\frac{49}{13}$ and $\beta = -\frac{28}{13}$

$\therefore\ x^{-\frac{49}{13}}y^{-\frac{28}{13}}$ is the required integrating factor.

Mutliplying equation (1) by I.F. *i.e.*, $x^{-\frac{49}{13}}y^{-\frac{28}{13}}$, we have

$$\left(2x^{-\frac{23}{13}}y^{-\frac{15}{13}} - 3x^{-\frac{49}{13}}y^{\frac{24}{13}}\right)dx + \left(3x^{-\frac{10}{13}}y^{-\frac{28}{13}} + 2x^{-\frac{36}{13}}y^{\frac{11}{13}}\right)dy = 0 \quad ...($$

Now equation (3) is exact. $\left[\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = -\frac{1}{13}\times\left(30x^{-\frac{23}{13}}y^{-\frac{28}{13}} + 72x^{-\frac{49}{13}}y^{\frac{11}{13}}\right.\right.$

Hence the solution is

$$\int_{y\ \text{constant}} \left(2x^{-\frac{23}{13}}y^{-\frac{15}{13}} - 3x^{-\frac{49}{13}}y^{\frac{24}{13}}\right)dx + 0 = c_1,$$

where c_1 is any arbitrary constant.

or $$\frac{-13}{10}\cdot 2x^{-\frac{10}{13}}y^{-\frac{15}{13}} + 3\cdot\frac{13}{36}x^{-\frac{36}{13}}y^{\frac{24}{13}} = c_1$$

or $$5x^{-\frac{36}{13}}y^{\frac{24}{13}} - 12x^{-\frac{10}{3}}y^{-\frac{15}{13}} = c.$$

EXERCISE 1.7

Solve the following differential equations :

1. $(y^3 - 2yx^2)\,dx + (2xy^2 - x^3)\,dy = 0.$
2. $(2x^2y^2 + y)\,dx - (x^3y - 3x)\,dy = 0.$ [M.D.U. 2013, 04,
3. $(y^2 + 2x^2y)\,dx + (2x^3 - xy)\,dy = 0.$
4. $(20x^2 + 8xy + 4y^2 + 3y^3)\,y\,dx + 4\,(x^2 + xy + y^2 + y^3)\,x\,dy = 0.$

ANSWERS

1. $x^2y^2(y^2 - x^2) = c$
2. $\frac{7}{5}\cdot x^{\frac{10}{7}}y^{-\frac{5}{7}} - \frac{7}{4}\cdot x^{-\frac{4}{7}}y^{-\frac{12}{7}} = c$
3. $6\sqrt{xy} - \left(\frac{y}{x}\right)^{3/2} = c$
4. $\left(4x^5 + 2x^4y + \frac{4}{3}x^3y^2 + x^3y^3\right)y = c.$